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LATERAL BUCKLING OF ECCENTRICALLY LOADED I-COLUMNS

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LATERAL BUCKLING OF ECCENTRICALLY LOADED I-COLUMNS

Mario G. Salvadori,[1] M. ASCE

SYNOPSIS

The critical combination of thrust and unequal end-moments or eccentricities (interaction curves) for the lateral buckling of columns elastically restrained against rotation about the principal axes at the end sections, and prevented from twisting about the column axis at the end sections are derived for various values of the Timoshenko parameter L^2/a^2 and of the end-moments ratio.

It is shown that for most practical cases these critical combinations are easily obtainable as soon as the critical value of equal end-moments without thrust is known for columns free to rotate at their ends (simply supported columns).

INTRODUCTION

The solution of the lateral stability problem for I-columns under eccentric thrust, i.e., under a combination of axial thrust and end-moments, consists in the determination of so-called interaction curves from which the critical value of the thrust may be derived for given end-moments or the critical value of the end-moments for a given thrust.

The shape of the interaction curves depends essentially on: 1) the geometric and strength characteristics of the I-column and, in particular, on the value of the Timoshenko parameter L^2/a^2 ; 2) the ratio r of the end-moments, or eccentricities; 3) the boundary conditions at the column ends.

Interaction curves for I-columns with equal end-moments were obtained by Johnston,(1)[2] for columns of rectangular cross-section with unequal end-moments by DiMaggio,(2) for I-columns with unequal end-moments by Salvadori.(3) All these solutions considered "simply supported" columns, whose ends are free to rotate around their principal axes at the ends and are prevented from twisting around the column axis there.

In the present paper interaction curves are obtained for I-columns whose ends are free to rotate in the plane of the web (strong plane), but are either elastically restrained or totally fixed in the plane of the flanges (weak plane) at the ends, while they are prevented from twisting around the column axis there. The consideration of unequal end-moments in the plane of the web, which may be due to continuity, allows the use of these interaction curves for columns elastically restrained in both the weak and the strong planes.

The treatment is limited to stresses within the elastic limit.

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2. Numbers in parentheses refer to the bibliography at the end of the paper.

List of Symbols

L = column length
ℓ = αL = reduced column length
h = column depth
B = minimum flexural rigidity of the column
C = torsional rigidity of the column without thrust
D = flexural rigidity of one flange in its own plane
M_2 = bending moment at top of column in the plane of the web
$M_{o,2}$ = bending moment at top of column in the plane of the web without thrust
M_1 = bending moment at the bottom of column in the plane of the web
r = M_1/M_2 = end-moment ratio
K_r = $M_2L/\sqrt{B\,C}$ = bending moment coefficient
$K_{o,r}$ = bending moment coefficient without thrust
$K_{o,1}$ = bending moment coefficient for equal end-moments without thrust
k_r = $M_2\ell/\sqrt{B\,C} = \alpha K_r$ = bending moment coefficient in terms of reduced length
$k_{o,1}$ = $\alpha K_{o,1}$ = reduced length bending moment coefficient for equal end-moments
μ_r = $K_{o,r}/K_{o,1}$ = magnification coefficient for unequal end-moments
$\bar{\mu}_{r_2}$ = minimum, safe value of μ_r
L^2/a^2 = $(2L^2C)/(h^2D)$ = Timoshenko parameter for I-beam lateral buckling
ℓ^2/a^2 = α^2L^2/a^2 = Timoshenko parameter in terms of reduced length
P = axial thrust
P_{cr} = π^2B/ℓ^2 = minimum Euler thrust
p = P/P_{cr} = thrust coefficient
M_{cr} = critical value of equal end-moments without thrust
$e_{2,1}$ = $M_{2,1}/P$ = top and bottom eccentricities
κ_r = $K_r/K_{o,r} = M_2/M_{o,2}$ = ratio of top moments with and without thrust
ϵ_2 = $e_2\sqrt{B/CL^2}$ = top eccentricity coefficient
P_T = CA/I_p = critical torsional thrust
A = cross-sectional area of column
I_p = polar moment of inertia of column cross-section
H = $C(1 - P/P_T)$ = reduced torsional rigidity
γ = P_{cr}/P_T = ratio of minimum Euler thrust to torsional critical thrust
P_E = Euler thrust in the plane of the web
V = total energy of buckled column
u = lateral displacement of buckled column in the plane of minimum rigidity
β = angular rotation of buckled column
z = abscissa along column axis with origin at bottom of column
$M(z)$ = $M_2\,[r + (1 - r)z/L]$ = bending moment in the plane of the column web
$u', u'', \ldots; \beta', \beta'', \ldots$ = derivatives of u and β with respect to z
η = $(u/h)\sqrt{2B/D}$ = non-dimensional lateral displacement
ϕ_n, ψ_n = functions of z for the expansion of η and β

Lateral Buckling under Equal End-Moments

A "simply supported" column of length L, depth h, minimum flexural rigidity B (in the plane of the flanges), torsional rigidity C, and flange rigidity D buckles laterally under the action of two equal end-moments M_s in the strong plane when:[4]

$$M_s = \frac{\pi\sqrt{BC}}{L}\sqrt{1 + \left(\frac{\pi}{L/a}\right)^2} \quad , \tag{1}$$

where:

$$\frac{L^2}{a^2} = \frac{2L^2C}{h^2\,D} \quad . \tag{2}$$

Introducing the moment coefficient:[3]

$$K_{o,1} = ML/\sqrt{BC} \quad , \tag{3}$$

its critical value is given by:

$$K_{o,1}\Big]_s = \pi\sqrt{1 + \left(\frac{\pi}{L/a}\right)^2} \quad . \tag{4}$$

When the I-column is completely fixed in the weak plane at the ends, the critical value of the end-moments is given by:[4]

$$M_f = \frac{2\pi\sqrt{BC}}{L}\sqrt{1 + \left(\frac{2\pi}{L/a}\right)^2} \quad , \tag{5}$$

and hence:

$$K_{o,1}\Big]_f = 2\pi\sqrt{1 + \left(\frac{2\pi}{L/a}\right)^2} \quad . \tag{6}$$

The critical value of $K_{o,1}$ may be expressed by means of a single formula if we introduce the concept of <u>reduced length</u> ℓ, i.e., of the length between inflection points when the same column buckles under thrust (Euler buckling). Letting

$$\ell = \alpha L \quad (1/2 \leq \alpha \leq 1) \quad , \tag{7}$$

the reduction coefficient α equals 1 for columns simply supported in the weak plane, 1/2 for columns fixed in the weak plane, and has values between 1/2 and 1 for columns elastically restrained in the weak plane. By means of ℓ the expression for $K_{o,1}$ becomes, whatever the conditions of restraint in the weak plane:

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3. The first subscript, o, indicates the absence of thrust; the second, 1, the equality of end-moments, whose ratio r equals one.

$$k_{o,1} = M\ell \;/\; \sqrt{BC} = \pi \sqrt{1 + \left(\frac{\pi}{\ell/a}\right)^2}\,, \tag{8}$$

and the critical value of the moments is given by:

$$M_{cr} = k_{o,1} \frac{\sqrt{BC}}{\alpha L}\ . \tag{9}$$

An analogous result was obtained by Mr. G. Winter, M. ASCE, for beams of rectangular cross-section.(7) Figure 1 gives the values of $k_{o,1}$ versus ℓ^2/a^2, which also appear in the row r = 1 of Table I, where ℓ = L and hence $k_{o,1} = K_{o,1}$.

Lateral Buckling under Unequal End-Moments

Indicating by:

$$r = M_{o,1}/M_{o,2} \tag{10}$$

the ratio of the moment $M_{o,1}$ at the bottom of the column to the moment $M_{o,2}$ at the top of the column, the critical values of $M_{o,2}$ and of $M_{o,1} = rM_{o,2}$ can be obtained by the Rayleigh-Ritz method as shown in the Appendix (Sec. 10). Tables I and II give:

$$K_{o,r} = M_{o,2}L/\sqrt{BC} \tag{11}$$

for columns simply supported in the weak plane (taken from ref.(3)) and for columns fixed in the weak plane, respectively, in terms of L^2/a^2 and r.

By means of the coefficients $K_{o,r}$ of Tables I and II it is easy to compute in terms of L^2/a^2 the ratios:

$$\mu_r = K_{o,r}/K_{o,1} = M_{o,2}/M_{o,2}\,]_{r\,=\,1} \tag{12}$$

of the critical moment $M_{o,2}$ at the top of the column to the critical value $M_{o,2}\,]\;r = 1$ of the same moment when $M_{o,2} = M_{o,1}(r = 1)$ for the two extreme conditions of simple support and fixity in the weak plane. Table III shows that for all practical purposes this ratio is independent of L^2/a^2 for $L^2/a^2 \leq 40$ and has the same value for both extreme conditions of support for r = 1/2 and r = 0. For r = -1/2 and r = -1 this ratio is slightly sensitive to the values of L^2/a^2 and to the conditions of support. Hence, it may be conjectured that for all values of r and for $L^2/a^2 \leq 40$ the ratio μ_r will be comprised between the extreme values corresponding to the conditions of simple support and of fixity (Fig. 2).

Noticing that in most practical cases $L^2/a^2 \leq 40$ and that errors of up to 10% are easily permissible if on the safe side, a simplified approximate

evaluation of the critical moments may be obtained by means of the minimum, safe, values $\bar{\mu}_r$ of μ_r given in Table IV or Fig. 2.

By means of Eqs. (8) and (11) the critical value of $M_{o,2}$ may be obtained, whatever the degree of restraint in the weak plane, by means of the formula:

$$M_{o,2} = \bar{\mu}_r \, M_{o,2}\Big]_{r=1} = \bar{\mu}_r \, k_{o,1} \frac{\sqrt{BC}}{\alpha L} , \tag{13}$$

where $k_{o,1}$ is given by Fig. 1. It is seen by comparison of Eq. (13) with Eq. (11) that:

$$K_{o,r} = \frac{\bar{\mu}_r}{\alpha} \, k_{o,1} . \tag{14}$$

As clearly shown by Eq. (14), the introduction of the (safe) ratios $\bar{\mu}_r$ and of the reduced length ℓ, reduces the solution of the lateral buckling problem under unequal end-moments, whatever the end restraints and the value of L^2/a^2, to the known solution of the buckling problem of a simply supported column in pure bending.

For example, an elastically restrained column of reduced length $\ell = 0.7\ L$ ($\alpha = 0.7$) with a moment $M_{o,2}$ at the top and no moment at the bottom ($M_{o,1} = rM_{o,2} = 0$, i.e., $r = 0$), for which $L^2/a^2 = 32$, has an $\ell^2/a^2 = (0.7\ L)^2/a^2 = 16$. Hence, from Fig. 1 or Table I, $k_{o,1} = 4.00$; from Fig. 2 or Table IV $\bar{\mu}_o = 1.82$, and by Eq. (13):

$$M_{o,2} = 1.82 \times 4.00 \frac{\sqrt{BC}}{0.7L} = 10.4 \sqrt{BC}/L . \tag{a}$$

A more accurate evaluation of $M_{o,2}$ may always be obtained by means of the values μ_r of Table III, but is seldom warranted in practice.

Lateral Buckling under Equal End-Moments and Thrust

Indicating by:

$$P_{cr} = \frac{\pi^2 B}{(\alpha L)^2} \tag{15}$$

the minimum critical value of the thrust in the absence of end-moments, and by:

$$M_{cr} = \pi \sqrt{1 + \left(\frac{\pi}{\alpha L/a}\right)^2} \; \frac{\sqrt{BC}}{\alpha L} \tag{16}$$

the critical value of equal end-moments in the absence of thrust, the critical combination of thrust P and equal end-moments M, that is the expression for the interaction curve P versus M, is given by:(1)

$$P/P_{cr} + (M/M_{cr})^2 = 1 \tag{17}$$

Letting:

$$p = P/P_{cr} = \frac{\alpha^2}{\pi^2} \frac{PL^2}{B} \quad , \tag{18}$$

and:

$$k_1 = M\ell / \sqrt{BC} \quad , \tag{19}$$

we obtain from Eq. (17):

$$k_1 = \pi \sqrt{1 + (\frac{\pi}{\ell/a})^2} \quad \sqrt{1 - p} \quad . \tag{20}$$

For $p = 0$, i.e., no thrust, $k_1 = k_{o,1}$ and the ratio:

$$k_1 / k_{o,1} = \sqrt{1 - p} \tag{21}$$

is independent of L^2/a^2 and α. The curve $k_1/k_{o,1}$ versus p is identical with the curve $K_o/K_{o,1}$ for $\alpha = 1$ and appears labeled $r = 1$ in Fig. 3 (thin curve).

By means of the values $k_{o,1} = K_{o,1}$ of Fig. 1 the critical value of equal end-moments with thrust is given by:

$$M = k_{o,1} \sqrt{1 - p} \; \frac{\sqrt{BC}}{\alpha L} \quad . \tag{22}$$

When the moments M are due to the eccentricity e of the thrust P, the critical value of e may be computed from the equation:

$$e = M/P \tag{23}$$

where M is given by Eq. (22).

Vice versa, Eq. (17) may be solved for P in terms of e, giving:

$$P = \frac{C}{2e^2} [-1 + \sqrt{1 + \frac{4\pi^2}{\alpha^2} \frac{Be^2}{CL^2}}] \quad . \tag{24}$$

The critical combinations of thrust P and of top moment M_2 for various values of the ratio $r = M_1/M_2$ and of L^2/a^2 may be computed by the methods explained in the Appendix for conditions of simple support (computed from ref.[3]), and for conditions of fixity in terms of the non-dimensional ratios $p = P/P_{cr}$ (Eq. (18)) and:

$$\kappa_r = K_r/K_{o,r} \tag{25}$$

where:

$$K_r = M_2 L/ \sqrt{BC} \ . \tag{26}$$

Remembering the definition of $K_{o,r}$ (Eq. (11)), it is also seen that:

$$\kappa_r = K_r/K_{o,r} = M_2/M_{o,2} \ . \tag{27}$$

Table V gives the largest percentage difference in absolute value between the ratios κ_r for simply supported and fixed ends, i.e., the values:

$$\Delta \kappa_r = \max. \left| \frac{\kappa_r]_s - \kappa_r]_f}{\kappa_r]_f} \right| \times 100 \ . \tag{28}$$

It is seen that $\Delta \kappa_r$ is less than 3% for all values of L^2/a^2 and of r, except for $r = -1$. Hence, for practical purposes, the interaction curves κ_r versus p are independent of L^2/a^2 and of the conditions of restraint in the weak plane, except for $r = -1$.

Figure 3 gives the graphs (thin curves) of the average value of κ_r versus p for $r = 1$, $r = 1/2$, $r = 0$, and $r = -1/2$. These graphs can be used whatever the value of L^2/a^2 and the end restraints. For $r = -1$, Fig. 3 gives 3 separate graphs for simple supported ends and 2 separate graphs for fixed ends in terms of L^2/a^2.

The discontinuity in the slope of the curves $r = -1$ of Fig. 3 corresponds to the meeting point of two separate branches of the interaction curves. The branch valid for lower values of κ_r represents a buckling deformation in which the flanges are displaced in the same direction; the branch valid for κ_r greater than 0.8 corresponds to flange displacements in opposite directions. This last branch if prolonged to meet the p axis, would cross it at a value corresponding to the second Euler load, i.e., at $p = 4$.

By means of the ratios κ_r of Eq. (27) the top moment M_2 may be written:

$$M_2 = \kappa_r M_{o,2} \ , \tag{29}$$

or by means of Eq. (13):

$$M_2 = \kappa_r \bar{\mu}_r k_{o,1} \frac{\sqrt{BC}}{\alpha L} . \qquad (30)$$

To illustrate the use of Fig. 3 consider a column restrained in the weak plane so as to have a reduced length $\ell = 0.7 L (\alpha = 0.7)$, with L^2/a^2 32, and loaded by a thrust $P = 0.6 P_{cr}$. Assume the restraint conditions in the strong plane to be such that $r = 0$. With $p = 0.6$ and $r = 0$ we obtain from Fig. 3 $\kappa_r = 0.66$. For $r = 0$ Table IV (or Fig. 2) gives $\bar{\mu}_o = 1.82$, for $\ell^2/a^2 = (0.7)^2 \times 32 = 16$ Table I (or Fig. 1) gives $k_{o,1} = 4.00$. Hence, by Eq. (30):

$$M_2 = \frac{0.66 \times 1.82 \times 4.00}{0.7} \frac{\sqrt{BC}}{L} = 6.87 \frac{\sqrt{BC}}{L} . \qquad (b)$$

The corresponding eccentricity e_2 of the thrust at the top of the column is given by:

$$e_2 = M_2/P = M_2/(0.6 P_{cr}) = \frac{6.87 \sqrt{BC}/L}{0.6\pi^2 B/(0.7L)^2} = 0.57 \sqrt{CL^2/B} \qquad (c)$$

To solve the instability problem for a given eccentricity:

$$e_2 = \epsilon_2 \sqrt{CL^2/B} , \qquad (31)$$

notice that $M_2 = P e_2 = p P_{cr} e_2$ and that by Eq. (26) $M_2 = K_r \sqrt{BC}/L$. Hence:

$$p P_{cr} e_2 = p \frac{\pi^2 B}{(\alpha L)^2} \epsilon_2 \sqrt{\frac{CL^2}{B}} = K_r \frac{\sqrt{BC}}{L} ,$$

from which:

$$K_r = (\pi/\alpha)^2 \epsilon_2 p , \qquad (32)$$

or, dividing both sides of this equation by $K_{o,r}$:

$$p = \frac{\alpha^2 K_{o,r}}{\pi^2 \epsilon_2} \kappa_r \ . \qquad (33)$$

The intersection of the straight line p versus κ_r of Eq. (33) with the p versus κ_r graph of Fig. 3 gives the critical value of p and hence of P.

For example, for $e_2 = 0.4 \sqrt{CL^2/B}$, i.e., $\epsilon_2 = 0.4$, $\alpha = 0.7$, $L^2/a^2 = 32$, $r = 0$, we have $\ell^2/a^2 = 16$ and by Eq. (14), Table IV, and Fig. 1 $K_{o,r} = \frac{\mu_r}{\alpha} k_{o,1} = 1.82 \times 4.00/0.7 = 14.8$. Hence:

$$\frac{\alpha^2 K_{o,r}}{\pi^2 \epsilon_2} = \frac{(0.7)^2 \times 14.8}{9.87 \times 0.4} = 1.83 \ .$$

The line through the origin of slope equal to 1.83 intersects the dashed curve r = 0 of Fig. 3 at p = 0.84; therefore, the critical value of the thrust equals:

$$P = 0.84 \ \pi^2 B/(0.7L) = 1.20 \ \pi^2 B/L \ . \qquad (d)$$

Reduction of Torsional Rigidity due to Thrust[4]

A column buckles in pure torsion under a thrust P_T uniformly distributed over its cross-section when:(5)

$$P_T = CA/ I_p \ , \qquad (34)$$

where A is the area of the cross-section and I_p its polar moment of inertia. The equivalent or reduced torsional rigidity H of the column under a thrust P is given by:(5,6)

$$H = C\ (1 - P/P_T) = C\ (1 - p\ P_{cr}/P_T) \ . \qquad (35)$$

In order to take into account the reduction of torsional rigidity in the evaluation of the interaction curves, the moment coefficient K_r is defined in terms of H rather than in terms of C:

$$K'_r = M_2 L/\sqrt{BH} = (M_2 L/\sqrt{BC})\ \sqrt{C/H} = K_r\ \sqrt{C/H} \ . \qquad (36)$$

4. This section is based on a suggestion of Mr. J. W. Clark.

Noticing that for $P = 0$ the reduced rigidity H_o equals C and indicating by $M_{o,2}$ the value of M_2 in the absence of thrust:

$$K'_{o,r} = M_{o,2}L/\sqrt{BH_o} = M_{o,2}L/\sqrt{BC} = K_{o,r} \; . \tag{37}$$

Hence the ratio of moments $M_2/M_{o,2}$ becomes:

$$M_2/M_{o,2} = (K_r/K_{o,r}) \sqrt{H/C} = \kappa_r \sqrt{1 - p\,(P_{cr}/P_T)} \; . \tag{38}$$

The ratio:

$$\gamma = P_{cr}/P_T = \frac{\pi^2 B\, I_p}{\alpha^2 L^2 CA} \tag{39}$$

varies, practically, between zero and one. When the reduction of torsional rigidity is negligible, $\gamma = 0$ and the thin curves of Fig. 3 remain unchanged. When the torsional rigidity is greatly reduced by the thrust, $\gamma = 1$ and the ordinates κ_r of these curves must be multiplied by $\sqrt{1 - p}$. The thick curves of Fig. 3 give κ_r versus p for $\gamma = 1$ and are on the safe side for all practical cases. For intermediate values of γ new curves may be easily computed.

To illustrate the influence of the reduction of torsional rigidity on the value of the critical moments, consider the column of the example of Sec. 5. Assuming the highest possible value of $\gamma (\gamma = 1)$ and $p = 0.6$, the moment M_2 of Eq. (b) becomes:

$$M_2 = \sqrt{1 - 0.6} \times 6.87 \sqrt{BC}/L = 4.36 \sqrt{BC}/L \; . \tag{e}$$

Influence of Deflections in the Plane of the Web

The interaction curves of the preceeding Sections are obtained under the assumption that the flexural rigidity of the column in the plane of the web is much greater than the rigidity in the plane of the flanges, so that bending deflections in the plane of the web may be neglected.

For the case of equal end-moments the influence of the deflections in the plane of the web may be taken into account by multiplying the critical bending moments M by the factor $1 - P/P_E$, where P_E is the Euler thrust in the plane of the web. Thus the curves $r = 1$ of Fig. 3 may be used provided:

$$\kappa_r/(1 - pP_{cr}/P_E) \tag{40}$$

be substituted for κ_r.

The influence of the deflections in the plane of the web decreases as r approaches -1. Hence this influence may be taken empirically into account, with sufficient accuracy, by substituting for κ_r the ordinates:

$$\kappa_r' = \kappa_r / [1 - \frac{1}{2} p (r + 1) P_{cr}/P_E] . \qquad (41)$$

In the examples of Secs. 5 and 6, assuming a ratio $P_{cr}/P_E = I_{min}/I_{max} = 0.2$, with r = 0 and p = 0.6 the critical moment becomes:

$$M_2 = 4.36(1-0.5 \times 0.6 \times 0.2) \sqrt{BC} /L = 4.10 \sqrt{BC} /L . \qquad (f)$$

CONCLUSIONS

The results of the preceding Sections prove that for all practical purposes the critical combinations of thrust and end-moments of I-columns elastically restrained in bending about the principal axes of their end-sections and fixed against twisting there, may be easily obtained once the critical value of equal end-moments for a simply supported column is known. By means of the graphs and tables presented in this paper the designer may evaluate critical thrusts and moments (or eccentricities) with a minimum of computations.

The most significant result achieved by this numerical research is perhaps the discovery of the practically linear relation between p and κ_r, valid for most practical cases when the reduction of torsional rigidity is taken into account.

Although it must be remembered that an extension of the present results to the inelastic range is essential to all engineering applications, it is hoped that their simplicity will encourage specification writers to the adoption of more accurate column formulas in order to achieve greater economy and safety in steel structures.

Mathematical Appendix

The values of the moment coefficients:

$$K_r = M_2 L / \sqrt{BC} \qquad (42)$$

and of the ratios $\kappa_r = K_r/K_{o,r}$ were obtained by the Rayleigh-Ritz method starting from the expression for the total energy V of an I-column buckled laterally under the action of thrusts P and end-moments:[3]

$$V = \frac{B}{2} \int_0^L (u'')^2 dz + \frac{Dh^2}{4} \int_0^L (\beta '')^2 dz$$
$$+ \frac{H}{2} \int_0^L (\beta ')^2 dz + \int_0^L M \beta u'' dz - \frac{P}{2} \int_0^L (u')^2 dz \qquad (43)$$

(see "List of Symbols" for meaning u, β, z and M).

The boundary conditions for columns simply supported in the weak plane are:

$$u(0) = u''(0) = u(L) = u''(L) = 0$$
$$\beta(0) = \beta''(0) = \beta(L) = \beta''(L) = 0 \tag{44}$$

For columns fixed at the ends in the weak plane the boundary conditions are:

$$u(0) = u'(0) = u(L) = u'(L) = 0$$
$$\beta(0) = \beta'(0) = \beta(L) = \beta'(L) = 0 \; . \tag{45}$$

The conditions of Eqs. (44) are satisfied by taking:

$$u(z) = \sum_{n=1}^{\infty} a_n \sin n\pi\, z/L \; ; \; \beta(z) = \sum_{n=1}^{\infty} b_n \sin n\pi\, z/L \; . \tag{46}$$

The corresponding Rayleigh-Ritz calculations are summarized in ref.[3]

To satisfy the conditions of Eqs. (45), let:

$$u(z) = h\sqrt{\frac{D}{2B}}\;\eta(z) \tag{47}$$

and expand η and β in the following series:

$$\eta(z) = \sum_{n=1}^{\infty} a_n\,\phi_n(z) + \sum_{n=1}^{\infty} c_n\,\psi_n(z) \tag{48}$$

$$\beta(z) = \sum_{n=1}^{\infty} b_n\,\phi_n(z) + \sum_{n=1}^{\infty} d_n\,\psi_n(z) \tag{49}$$

where:

$$\phi_n(z) = \frac{1}{4n^2\pi^2}\,(1 - \cos 2n\pi\, z/L) \tag{50}$$

$$\psi_n(z) = \frac{1}{4\nu^2}\,[2z/L - 1 - \sin \nu_n(2z/L - 1)/\sin \nu_n] \tag{51}$$

and ν_n are roots of the transcendental equation:

$$\tan \nu = \nu. \tag{52}$$

The choice of these functions greatly simplifies the calculations, since they are orthogonal together with their first and second derivatives in almost all the combinations appearing in the integrals to be evaluated in minimizing the total energy V.

In order to obtain an accuracy of 1% or better the cases $r = 1/2$, $r = 0$, $r = -1/2$, and $r = -1$ were solved taking into account one term of each series and from the corresponding values K_r were computed the ratios $\kappa_r = K_r/K_{o,r}$. The corresponding determinantal equations are of the 4th order. The case $r = -1$ was then solved with two terms of each series, leading to determinantal equations of the 8th order. The new K_r differed substantially from those of the first approximation, but the new ratios κ_r were instead practically identical with those obtained by means of the first approximation. Hence the ratios κ_r of the first approximation were considered correct within the required accuracy and only the $K_{o,r}$ were computed again by means of 2 terms of each series for $r = 1/2$, $r = 0$, and $r = -1/2$, and with 2 and 3 terms of each series for $r = -1$. In this last case the determinantal equations of order 8 and 12 split, for $p = 0$, into two separate equations of order 4 and 6, whose roots correspond to the two branches of the curves $r = -1$ of Fig. 3. The intersection of the two branches was also determined in order to draw the graphs with greater accuracy.

The extremely complicated calculations were carried out by the staff of the Istituto Nazionale per le Applicazioni del Calcolo, the mathematical laboratory of the Italian National Research Council directed by Professor Mauro Picone. Professor Aldo Ghizzetti, Vice-Director of the Institute, and Dr. W. Gross supervised the calculations. Mrs. E. D. Broadbent, B.Sc., Dr. C. Mengotti Marzolla, and Mr. A. Perugini were in charge of the calculations, Mr. V. Ceccarini participated in the calculations and drew the graphs.

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values of $K_{0,r} = K_r]_{p=0}$ (Simple support; $l = L$)

r \ l^2/a^2	0.1	1	2	6	10	16	24	32	40	100	∞
1.0	31.40	10.36	7.66	5.11	4.43	4.00	3.73	3.59	3.51	3.29	3.14
0.5	41.40	13.68	10.11	6.75	5.84	5.27	4.92	4.73	4.62	4.33	4.12
0	58.30	19.22	14.19	9.44	8.16	7.33	6.83	6.55	6.38	5.94	5.56
-0.5	82.39	27.14	20.00	13.24	11.40	10.18	9.42	9.00	8.74	8.00	7.32
-1.0	86.07	28.40	20.98	13.95	12.07	10.85	10.09	9.69	9.43	8.72	8.03

Table I

values of $K_{o,r} = K_r \big]_{p=0}$ (Fixed ends)

r \ L^2/a^2	0.1	1	2	6	10	16	24	32	40	100	∞
1.0	125.00	39.98	28.61	17.30	13.98	11.70	10.22	9.39	8.86	7.42	6.28
0.5	164.98	52.76	37.76	22.82	18.44	15.43	13.47	12.38	11.67	9.76	8.19
0	230.79	73.79	52.80	31.89	25.74	21.52	18.77	17.22	16.22	13.50	11.20
-0.5	318.97	101.93	72.91	43.94	35.40	29.52	25.66	23.48	22.06	18.10	14.40
-1.0	322.02	102.90	73.59	44.34	35.71	29.76	25.85	23.63	22.18	18.11	14.00

Table II

values of $\mu_r = K_{0,r}/K_{0,1}$

$\frac{L^2}{a^2}$	$r = 1$		$r = 0.5$		$r = 0$		$r = -0.5$		$r = -1$	
	supp.	fix.	supp.	fix.	supp.	fix.	supp.	fix.	supp.	fix.
0.1	1.00	1.00	1.32	1.32	1.86	$1.8\dot{5}$	2.62	2.55	2.74	2.58
1	1.00	1.00	1.32	1.32	1.86	$1.8\dot{5}$	2.62	$2.5\dot{5}$	2.74	2.57
2	1.00	1.00	1.32	1.32	1.85	$1.8\dot{5}$	2.61	$2.5\dot{5}$	2.74	2.57
6	1.00	1.00	1.32	1.32	$1.8\dot{5}$	1.84	2.59	2.54	2.73	2.56
10	1.00	1.00	1.32	1.32	1.84	1.84	2.57	2.53	2.72	2.55
16	1.00	1.00	1.32	1.32	1.83	1.84	$2.5\dot{5}$	2.52	2.71	2.54
24	1.00	1.00	1.32	1.32	1.83	1.84	2.53	2.51	2.71	2.53
32	1.00	1.00	1.32	1.32	1.82	1.83	2.51	2.50	2.70	2.52
40	1.00	1.00	1.32	1.32	1.82	1.83	2.49	2.49	2.69	2.50
100	1.00	1.00	1.32	1.32	1.81	1.82	2.43	2.44	2.65	2.44
∞	1.00	1.00	1.31	1.30	1.77	1.78	2.33	2.29	2.56	2.23

Table III

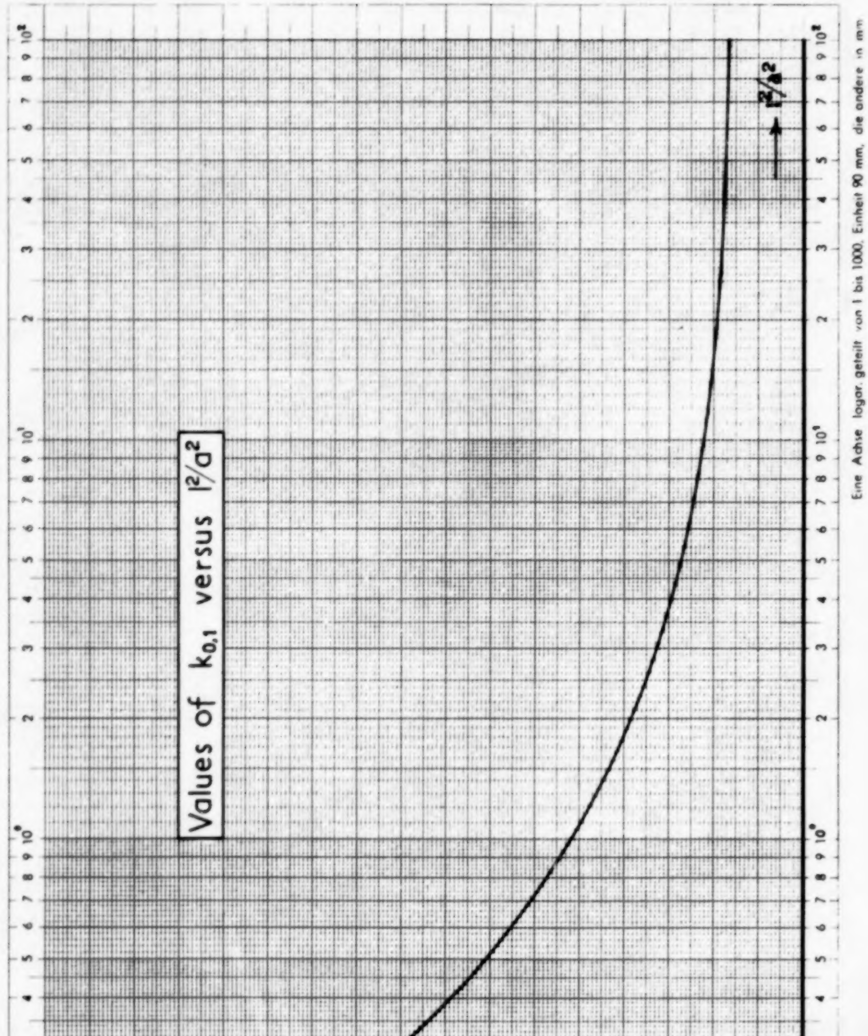
r	1	0.5	0	−0.5	−1
$\bar{\mu}_r$	1.00	1.32	1.82	2.49	2.50

Table IV

$$\max \left| \frac{(K_r/K_{o,r})\text{supp.} - (K_r/K_{o,r})\text{fix.}}{(K_r/K_{o,r})\text{fix.}} \right| 100$$

r \ p	0	0.2	0.4	0.6	0.8	1
1.0	0	0.00	0.13	0.16	0.00	0
0.5	0	0.22	0.26	0.31	0.66	0
0	0	0.55	0.50	0.30	0.42	0
-0.5	0	1.40	2.00	1.63	0.50	0
-1.0	0	2.73	5.13	9.62	14.45	0

Table V



Values of $k_{0,1}$ versus l^2/a^2
$k_{0,1}$
l^2/a^2
32
28
24
20
16
12
8
4
0
Eine Achse logar. geteilt von 1 bis 1000, Einheit 90 mm, die andere in mm

Fig. 1

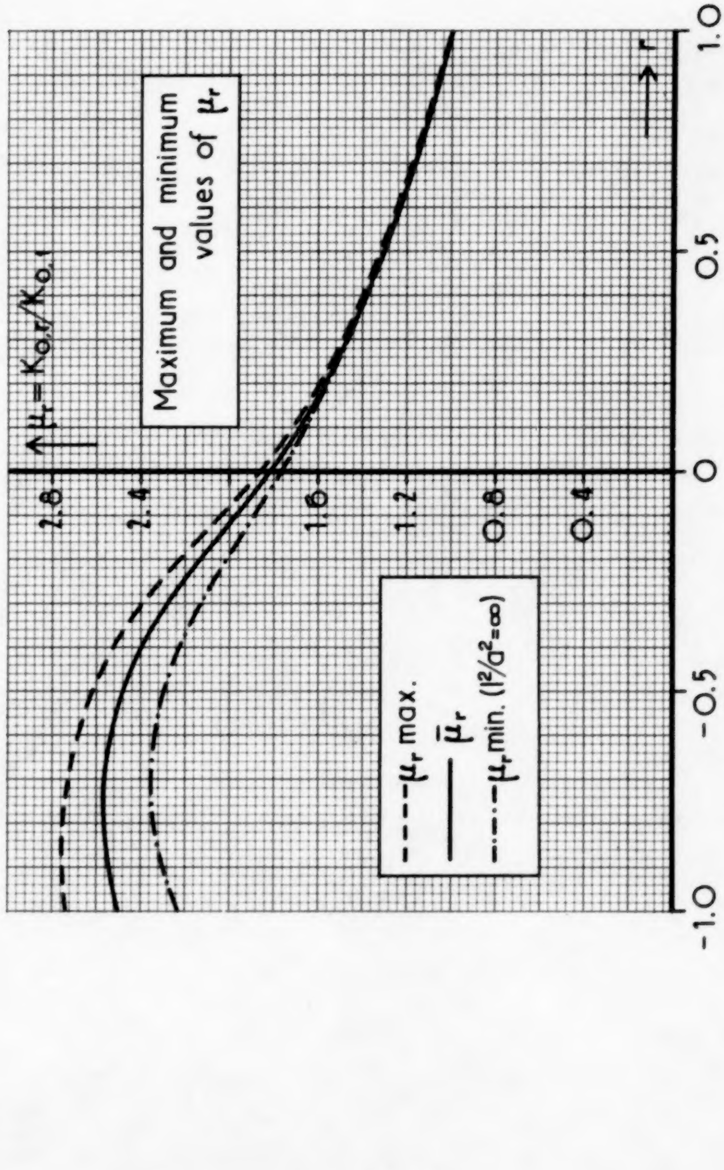
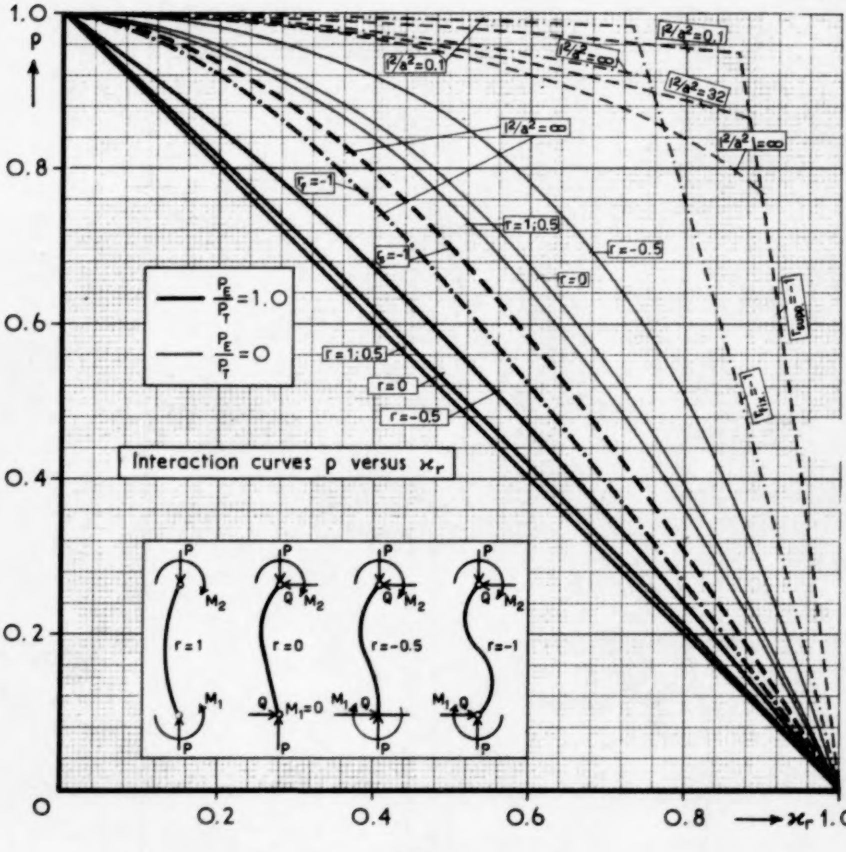

$\mu_r = K_{o,r}/K_{o,1}$
Maximum and minimum values of μ_r
μ_r max.
$\bar{\mu}_r$
μ_r min. $(l^2/a^2=\infty)$
2.8
2.4
1.6
1.2
0.8
0.4
-1.0
-0.5
0
0.5
1.0
r

Fig. 2



1.0
p
0.8
0.6
0.4
0.2
0
0.2
0.4
0.6
0.8
→ $\varkappa_r$ 1.0
$l^2/a^2=0.1$
$l^2/a^2=\infty$
$l^2/a^2=32$
$r_f=-1$
$r=1;0.5$
$r=-0.5$
$r=0$
$r_s=-1$
$r_{supp}=-1$
$r_{fix}=-1$
$\frac{P_E}{P_T}=1.0$
$\frac{P_E}{P_T}=0$
Interaction curves p versus $\varkappa_r$
P
Q
M_1
M_2
$M_1=0$
r=1
r=0
r=-0.5
r=-1

Fig. 3